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Calculation of Expected Depletion Time When Demand is Stuttering Poisson

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8

CALCULATION OF EXPECTED DEPLETION TIME WHEN DEMAND IS STUTTERING POISSON

Consider an inventory system in which there is initial stock of an item and for which there is random demand. It is often important to know the depletion time of the stock given that no re-supply occurs in the interim. Because the demand for the item occurs at random, the depletion time is also random. However, it does have a mean value. This paper presents the derivation of the expected depletion time when demand obeys the stuttering Poisson probability law. The derivation is greatly simplified by the application of an infrequently used formula for the calculation of the expected value of a random variable.

The stuttering Poisson distribution is one of many in the class of compound Poisson distributions, and results from exponentially distributed arrivals which have order sizes governed by a geometric distribution. It has been found to have several applications in the study of real world systems. In particular, studies for the Army have shown the stuttering Poisson probabilities to be a good approximation for the frequency of demand in a given period. The distribution, therefore, has been used in several inventory models developed for the Army for use in supply control decisions. The need to know the expected depletion time of stock for one of these models led to the writing of this paper.

Letting the notation P(j, t) mean the probability that demand equals j in a period of length t, the P(j, t) are[1]

$$P(0,t) = e^{-\lambda t}$$

(1) $$P(1,t) = \frac{\lambda t}{S} e^{-\lambda t}$$

$$P(j,t) = \sum_{i=1}^{j} \frac{e^{-\lambda t}}{i!} (\lambda t)^i \binom{j-1}{j-i} \left(\frac{S-1}{S}\right)^{j-i} \left(\frac{1}{S}\right)^i \quad \text{for} \quad j = 2, 3, \ldots$$

where $\frac{1}{\lambda}$ is the mean inter-arrival time of orders, and S is the mean order size.

If N is the value of stock on hand at time zero, then the probability that the depletion time, T, is greater than t is exactly the probability that fewer than N items are demanded in the time t. Thus,

$$\mathrm{Prob}[T > t] = \sum_{j=0}^{N-1} P(j,t)$$

Using the relationship for the calculation of the expected value of a random variable[2]

(2) $$E(x) = \int_0^\infty \mathrm{Prob}[X > x]\, dx$$

we have that the expected depletion time E(T) is

(3) $$E(T) = \int_0^\infty e^{-\lambda t} dt + \int_0^\infty \frac{\lambda t}{S} e^{-\lambda t} dt + \sum_{j=2}^{N-1} \sum_{i=1}^{j} \frac{e^{-\lambda t} (\lambda t)^i}{i!} \binom{j-1}{j-i} \left(\frac{S-1}{S}\right)^{j-i} \left(\frac{1}{S}\right)^i dt$$

References:

1. Galiher, H. P., "Ordnance Logistics Studies - II Secondary Item Supply Control", Interim Technical Report No. 9, Fundamental Investigations in Operations Research, MIT, June 1957.

2. Morse, Philip M., Queues, Inventories and Maintenance, John Wiley and Sons, Inc., New York, 1958.

(4) $$E(T) = \frac{1}{\lambda} + \frac{1}{\lambda S} + \frac{1}{\lambda}\sum_{j=2}^{N-1}\sum_{i=1}^{j}\binom{j-1}{j-i}\left(\frac{S-1}{S}\right)^{j-i}\left(\frac{1}{S}\right)^{i}$$

If each term in the double summation is multiplied by $\frac{j}{j} \times \frac{i}{i}$ then (4) becomes

(5) $$E(T) = \frac{1}{\lambda} + \frac{1}{\lambda S} + \frac{1}{\lambda}\sum_{j=2}^{N-1} \frac{1}{j}\sum_{i=1}^{j} i\binom{j}{i}\left(\frac{S-1}{S}\right)^{j-i}\left(\frac{1}{S}\right)^{i}$$

Thus,

(6) $$E(T) = \frac{1}{\lambda} + \frac{1}{\lambda S} + \sum_{j=2}^{N-1} \frac{1}{\lambda S}$$

Finally,

(7) $$E(T) = \frac{N+S-1}{\lambda S}$$

As is apparent, the derivation of E(T) requires only simple mathematics. However, this simplicity can be attributed only to the use of equation (2) in place of the more common

$$E(X) = \int_{-\infty}^{\infty} x\,p(x)\,dx$$

The equation (2) eliminates many steps because it was able to use a simpler expression than the density function.

It is felt that many situations occur in calculations where the use of (2) will simplify the required effort. When the random variable is continuous, the logical derivation of probabilities must often be made with respect to

the di tribution function and not the density function. More often than not, the furth derivation of the density results in a more complicated expression. Therefore, (2) might often be the best formula to use for the calculation of expected values.

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13. ABSTRACT

A derivation of the expected depletion time of an initial value of stock is presented when demand is governed by the stuttering Poisson distribution. The derivation is simplified by the application of an infrequently used formula for the calculation of the expected value of a random variable.

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